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Strong Unicity of Best Uniform Approximations from Periodic Spline Spaces*

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In this paper we give a complete characterization of the strongly unique best uniform approximations from periodic spline spaces. We distinguish between even-dimensional and odd-dimensional periodic spline spaces. These spaces are weak Chebyshev if and only if their dimension is odd. We show that the strongly unique best approximation from periodic spline spaces of odd dimension can be characterized alone by alternation properties of the error. This is not the case for even dimension. In this case an additional interpolation condition arises in our characterization. © 1999 Academic Press

INTRODUCTION

Standard spaces for approximating $(b-a)$-periodic, continuous functions $f: \mathbb{R} \mapsto \mathbb{R}$, (i.e., $f(x) = f(x + (b-a))$, $x \in \mathbb{R}$) are spaces of periodic splines. We denote by $P_m(K_n)$ the n-dimensional space of $(b-a)$-periodic splines of degree $m \geqslant 1$ with the set of knots $K_n = \{x_0, ..., x_n\}$, where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Moreover, we denote $C = \{f \in C(\mathbb{R}) : f$ is $(b-a)$-periodic$\}$.

In this paper we treat best approximation by $P_m(K_n)$ with respect to the uniform norm, defined by

$$\|f\|_\infty = \sup\{|f(t)| : t \in [a, b]\}, \qquad f \in C$$

i.e., to determine $p_f \in P_m(K_n)$ such that

$$\|f - p_f\|_\infty \leqslant \|f - p\|_\infty, \qquad p \in P_m(K_n).$$

* This paper contains results on approximation by periodic splines from the author's dissertation [23], written under the supervision of G. Walz at the University of Mannheim, Germany.

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Best uniform approximations from $P_m(K_n)$ are not unique, in general.

Davydov [4] gave some sufficient conditions for unicity of best uniform approximation from $P_m(K_n)$.

Meinardus and Nürnberger [12] showed that every function $f \in C$ has a unique L_1-approximation p_f from $P_m(K_n)$ (i.e., $\int_a^b |(f-p_f)(t)|\, dt < \int_a^b |(f-p)(t)|\, dt$, $p \in P_m(K_n) \backslash \{p_f\}$). This result holds independent of the dimension.

In this paper we consider the so-called strongly unique best uniform appproximation $p_f \in P_m(K_n)$ of a given $f \in C$, i.e.,

$$\|f-p_f\|_\infty + K_f \|p-p_f\|_\infty \leqslant \|f-p\|_\infty, \qquad p \in P_m(K_n)$$

where $K_f > 0$. For weak Chebyshev spaces (in particular spline spaces without periodicity), Nürnberger [15] found a complete characterization of the strongly unique best uniform appproximation. The aim of this paper is to give a complete characterization of the strongly unique best uniform appproximations from periodic spline spaces. The characterization is different for odd and even dimension of $P_m(K_n)$.

As usual, results on interpolation play an essential role for treating approximation problems. The interpolation problem with respect to $\{t_1, ..., t_n\}$ so that $t_1 < \cdots < t_n < t_1 + (b-a)$ is to determine a periodic spline $p \in P_m(K_n)$ such that

$$p(t_k) = f(t_k), \qquad k = 1, ..., n \tag{1}$$

where $f \in C$ is given.

Schumaker [18; 19, Theorem 8.8] showed that if the dimension of $P_m(K_n)$ is odd, then a Schoenberg-Whitney type condition characterizes those sets $\{t_1, ..., t_n\}$ for which the interpolation problem (1) has a unique solution from $P_m(K_n)$ for every $f \in C$. If the dimension of $P_m(K_n)$ is even, then this condition is only necessary.

Davydov [2] found a characterization of best uniform approximations from $P_m(K_n)$. In the case of odd dimension, this characterization is similar to the classical result of Rice [17] and Schumaker [20] on best uniform approximations from spline spaces (without periodicity) (An alternative proof was given in [23, Satz 4.1.1].) Moreover, in the case of even dimension, an additional condition appears in the characterization given in [2].

The following investigations show that the case of odd dimension is similar to the characterization of Nürnberger [16, Theorem 4.4.] on strongly unique best uniform approximation from spline spaces (without periodicity). On the other hand, if the dimension of $P_m(K_n)$ is even, then an additional interpolation condition is needed in our characterization theorem.

To prove these results, we give some statements on (Hermite-) interpolation by periodic spline spaces of even dimension which are of independent interest.

This paper is organized as follows. In the first section we state our main results (Theorems 3 and 5) on the strongly unique best uniform approximations from periodic spline spaces. Proofs of these characterizations are given in Section 4 (Theorem 3) and Section 5 (Theorem 5). In Section 2, we give necessary conditions for the strongly unique best uniform approximation from $P_m(K_n)$ which can be formulated independent of the dimension.

Section 3 contains some preliminary results which are needed for the proofs in Sections 4 and 5. Among others, some results on interpolation by periodic splines in the case of even dimension are given there.

1. MAIN RESULTS

Let m, n be natural numbers and $K_n = \{x_0, ..., x_n\}$ be a set of knots so that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Moreover, we set $x_{i+jn} = x_i + j(b-a)$, $i = 1, ..., n$, $j \in \mathbb{Z} \backslash \{0\}$. We write $C^q(\mathbb{R})$ for the space of all q times continuously differentiable, real functions and Π_m for the space of polynomials of degree at most m. We denote by

$$S_m = \{s \in C^{m-1}(\mathbb{R}) : s|_{[x_i, x_{i+1}]} \in \Pi_m, i \in \mathbb{Z}\}$$

and call

$$P_m(K_n) = S_m \cap C$$

the space of periodic splines of degree m with the set of knots K_n. Each spline $s \in S_m$ can be written as $s \equiv \sum_{k=-\infty}^{\infty} \alpha_k B_k$, where B_k, $k \in \mathbb{Z}$, is the B-Spline with support $[x_k, x_{k+m+1}]$.

We call an n-dimensional subspace G of C weak Chebyshev, if every function $g \in G$ has at most $n-1$ sign changes in each period, i.e., there does not exist a set $\{t_1, ..., t_{n+1}\}$ such that $t_1 < \cdots < t_{n+1} < t_1 + (b-a)$, with $g(t_i)\, g(t_{i+1}) < 0$, $i = 1, ..., n$. It is well known (cf. Nürnberger [16, Theorem 1.6.]) that the weak Chebyshev property of G is equivalent to the existence of a basis $\{g_1, ..., g_n\} \subseteq G$ such that for all $t_1 < \cdots < t_n < t_1 + (b-a)$,

$$D\begin{pmatrix} g_1 & \cdots & g_n \\ t_1 & \cdots & t_n \end{pmatrix} = \det \begin{pmatrix} g_1(t_1) & \cdots & g_n(t_1) \\ \vdots & & \vdots \\ g_1(t_n) & \cdots & g_n(t_n) \end{pmatrix} \geqslant 0.$$

We note that by induction on m and using Rolle's theorem it is not difficult to verify that every spline in $P_m(K_n)$ has at most $n-1$ (respectively n) sign changes over period, if n is odd (respectively even). In particular, the n-dimensional space $P_m(K_n)$ is weak Chebyshev if and only if n is odd. This result is connected with the fact that each spline $p \in P_m(K_n)$ with only finitely many zeros in $[a, b)$ has at most $n-1$ (respectively n) zeros (counting multiplicities) in $[a, b)$, if n is odd (respectively even). Therefore, if n is odd and $n \leqslant m+1$, then $P_m(K_n)$ restricted to $[a, b)$, is a Chebyshev space (cf. Meinardus and Walz [13] and [23, Satz 1.1.5]).

Following Davydov [2], we give the following definition for the case of even dimension.

DEFINITION 1. Let n be even. If there exists a spline from $P_m(K_n)$ having exactly the simple zeros $t_1 < \cdots < t_n (< t_1 + (b-a))$ in $[t_1, t_1 + (b-a))$, then we call $\{t_1, ..., t_n\}$ a not-interpolation set, briefly denoted by NI-set, for $P_m(K_n)$.

The characterization of NI-sets for $P_1(K_n)$ was given in [10] (see also [23, Satz 2.2.8]). Moreover, examples of NI-sets for periodic spline spaces of higher degree can be found in [7, 11, 14, 21, 23].

We begin with the definition of alternating extreme points for periodic functions, which play a fundamental role in the following investigations.

DEFINITION 2. Let $h \in C$, $\xi \in [a, b)$ and a non-empty subset $M \subseteq [\xi, \xi + (b-a)]$ be given. Let

$$E_{h, M} = \{t \in M \colon |h(t)| = \|h\|_\infty\}$$

denote the set of extremal points of h with respect to M. Points $t_1 < \cdots < t_r (< t_1 + (b-a))$ in M are called *alternating extreme points*, briefly denoted by $\mathscr{A}$-points, of h in M if there exists a sign $\sigma \in \{-1, 1\}$ such that

$$\sigma(-1)^k \, h(t_k) = \|h\|_\infty, \qquad k = 1, ..., r.$$

We briefly denote $\{t_1, ..., t_r\}$ an $\mathscr{A}$-set of h in M. For the maximal number of $\mathscr{A}$-points of h in M, we write $\mathscr{A}(h)|_M$.

The following result shows that if the periodic spline space is of odd dimension, then the strongly unique best uniform approximation is characterized alone by alternation properties of the error. (We remark that it is non-trivial to deduce this result from Nürnberger [15].)

THEOREM 3. *Suppose that n is odd. Let $f \in C \backslash P_m(K_n)$ and a spline $p_f \in P_m(K_n)$ be given. The following statements are equivalent.*

(i) *The spline p_f is a strongly unique best uniform approximation of f from $P_m(K_n)$.*

(ii) *For every interval (x_i, x_{i+m+j}), $j = 1, ..., n-m-1$, $i \in \mathbb{Z}$, we have*

$$\mathscr{A}(f - p_f)|_{(x_i, x_{i+m+j})} \geqslant j+1 \tag{2}$$

if $n > m+1$, and

$$\mathscr{A}(f - p_f)|_{[a, b]} \geqslant n+1. \tag{3}$$

We note that the result of Theorem 3 is similar to the characterization of the strongly unique best uniform approximation from the spline space

$$S_m(x_1, ..., x_{n-1}) = \{s = s_0|_{[a, b]} : s_0 \in S_m\}$$

(without periodicity) given by Nürnberger [15; 16, Theorem 4.4].

The main purpose of this paper is to characterize the strongly unique best uniform approximation from $P_m(K_n)$ in the case of even dimension. We need the following definition.

DEFINITION 4. Let $h \in C$, $\xi \in [a, b)$ and a non-empty interval $I \subseteq [\xi, \xi + (b-a)]$ be given. Moreover, let $\{t_1, ..., t_r\}$ be an $\mathscr{A}$-set of h in I, where $r = \mathscr{A}(h)|_I$, and set $t_0 = \inf\{t : t \in I\}$, $t_{r+1} = \sup\{t : t \in I\}$. We define for all $k \in \{1, ..., r\}$,

$$\alpha_k = \min\{t \in [t_{k-1}, t_k] : h(t) = h(t_k)\},$$

$$\beta_k = \max\{t \in [t_k, t_{k+1}] : h(t) = h(t_k)\},$$

and call $I_k = [\alpha_k, \beta_k]$, $k = 1, ..., r$, the *alternation intervals* of h in I. We denote by $I_k^0 = (\alpha_k, \beta_k)$.

We now state our characterization. The main difference to Theorem 3 is that it contains an additional interpolation condition for a set placed inside the alternation intervals (of an interval with length $b-a$) of the error.

THEOREM 5. *Suppose that n is even. Let $f \in C \backslash P_m(K_n)$ and a spline $p_f \in P_m(K_n)$ be given. The following statements are equivalent.*

(i) *The spline p_f is a strongly unique best uniform approximation of f from $P_m(K_n)$.*

(ii) *For every interval (x_i, x_{i+m+j}), $j = 1, ..., n-m-1$, $i \in \mathbb{Z}$, we have*

$$\mathscr{A}(f - p_f)|_{(x_i, x_{i+m+j})} \geqslant j+1 \tag{4}$$

if $n > m+1$, and there exists a $\xi \in [a, b)$ such that

$$\mathscr{A}(f - p_f)|_{[\xi, \xi + (b-a)]} \geqslant n+1 \tag{5}$$

and if there exists $\xi^* \in [a, b)$ such that $f - p_f$ has exactly n alternation intervals I_k, $k = 1, ..., n$, in $[\xi^*, \xi^* + (b-a)]$, then there exists a NI-set $\{t_1, ..., t_n\}$ for $P_m(K_n)$ such that $t_k \in I_k$, $k = 1, ..., n$, and $t_k \in I_k^0$ whenever I_k^0 is non-empty.

For completeness, we formulate the characterization of best uniform approximations from $P_m(K_n)$ due to Davydov.

THEOREM 6 (Davydov [2]). *Let $f \in C \backslash P_m(K_n)$ and a spline $p_f \in P_m(K_n) = span\{p_1, ..., p_n\}$ be given. Consider the following statements.*

(i) *The spline p_f is a best uniform approximation of f from $P_m(K_n)$.*

(ii) *There exist $j \in \{1, ..., n\}$, $i \in \mathbb{Z}$, such that*

$$\mathscr{A}(f - p_f)|_{[x_i, x_{i+j}]} \geqslant d+1$$

where $d = dim(P_m(K_n)|_{[x_i, x_{i+j}]})$.

(iii) *There exists an $\mathscr{A}$-set $\{t_1, ..., t_n\}$ of $f - p_f$ in $[a, b]$, which is a NI-set for $P_m(K_n)$.*

(iv) *There exist $\xi \in [a, b)$ and an $\mathscr{A}$-set $\{t_1, ..., t_{n+1}\}$ of $f - p_f$ in $[\xi, \xi + (b-a))$ with*

$$D\begin{pmatrix} p_1 & \cdots & p_n \\ t_1 & \cdots & t_n \end{pmatrix} D\begin{pmatrix} p_1 & \cdots & p_n \\ t_2 & \cdots & t_{n+1} \end{pmatrix} > 0.$$

If n is odd, then (i) *and* (ii) *are equivalent. If n is even, then* (i) *holds if and only if* (ii), *with $j \leqslant n-m$, if $n \geqslant m+1$, or* (iii) *or* (iv) *is satisfied.*

2. NECESSARY CONDITIONS FOR STRONG UNICITY

In this section we give necessary conditions for the strongly unique best approximation from $P_m(K_n)$, which can be formulated independently of the dimension. For this, we need a lemma on the existence of certain functions from weak Chebyshev spaces and the following characterization which follows from Wulbert [22] (see also Bartelt and McLaughlin [1]).

THEOREM 7. *Let $f \in C \backslash P_m(K_n)$ and a spline $p_f \in P_m(K_n)$ be given. The following statements are equivalent.*

(i) *The spline* p_f *is a strongly unique best uniform approximation of* f *from* $P_m(K_n)$.

(ii) *There does not exist a non-trivial spline* $p \in P_m(K_n)$ *such that*

$$(f - p_f)(t)\, p(t) \geqslant 0, \qquad t \in E_{f-p_f,\,[a,b]}. \tag{6}$$

The next lemma follows from a well-known result on weak Chebyshev spaces (Jones and Karlovitz [6], Deutsch, *et al.* [5], see also Nürnberger [16, Corollary 1.7.]).

LEMMA 8. *The following statements hold.*

(i) *Let* $S = span\{B_i, ..., B_{i+j-1}\}$. *Then for all integers* $r \in \{1, ..., j\}$ *and all points* $x_i = t_1 < t_2 < \cdots < t_r < t_{r+1} = x_{i+m+j}$, *there exists a non-trivial spline* $s \in S$ *such that*

$$(-1)^k s(t) \geqslant 0, \qquad t \in [t_k, t_{k+1}], \quad k = 1, ..., r.$$

(ii) *Suppose that* n *is odd. Then for all integers* $r \in \{1, ..., n\}$ *and all points* $t_1 < t_2 < \cdots < t_r < t_{r+1} = t_1 + (b - a)$, *there exists a non-trivial spline* $p \in P_m(K_n)$ *such that*

$$(-1)^k p(t) \geqslant 0, \qquad t \in [t_k, t_{k+1}], \quad k = 1, ..., r.$$

The following lemma gives necessary conditions for the strongly unique best uniform approximation from $P_m(K_n)$, which are independent of the dimension.

LEMMA 9. *Let* $f \in C \backslash P_m(K_n)$ *and a spline* $p_f \in P_m(K_n)$ *be given. If* p_f *is a strongly unique best uniform approximation of* f *from* $P_m(K_n)$, *then the following statements hold*:

(i) *For every interval* (x_i, x_{i+m+j}), $j = 1, ..., n - m - 1$, $i \in \mathbb{Z}$, *we have*

$$\mathscr{A}(f - p_f)\,|_{(x_i,\, x_{i+m+j})} \geqslant j + 1 \tag{7}$$

if $n > m + 1$.

(ii) *There exists a* $\xi \in [a, b)$ *such that*

$$\mathscr{A}(f - p_f)\,|_{[\xi,\, \xi + (b-a)]} \geqslant n + 1. \tag{8}$$

Proof. We first show (ii). We begin with the case that n is odd. Suppose, contrary to our claim, that

$$\mathscr{A}(f - p_f)\,|_{[\xi,\, \xi + (b-a)]} \leqslant n$$

for any choice of $\xi \in [a, b)$. In particular,

$$\mathscr{A}(f-p_f)|_{[a,b]} = r \leqslant n. \tag{9}$$

We choose an $\mathscr{A}$-set $\{t_1, ..., t_r\}$ of $f-p_f$ in $[a, b]$. Thus,

$$\sigma(-1)^k (f-p_f)(t_k) = \|f-p_f\|_\infty, \qquad k=1, ..., r$$

where $\sigma \in \{-1, 1\}$. By definition of the accompanying alternation intervals $I_k = [\alpha_k, \beta_k]$, $k=1, ..., r$, of $f-p_f$ in $[a, b]$ (see Definition 4), we have

$$\sigma(-1)^k (f-p_f)(t) = \|f-p_f\|_\infty, \qquad t \in E_{f-p_f, [a,b]} \cap I_k, \quad k=1, ..., r \tag{10}$$

and by (9)

$$E_{f-p_f, [a,b]} \subseteq \bigcup_{k=1}^{r} I_k. \tag{11}$$

Set $t_1^* = a$, $t_{r+1}^* = b$ and choose $t_k^* \in (\beta_{k-1}, \alpha_k)$, $k=2, ..., r$. Since $r \leqslant n$, by Lemma 8, (ii), there exists a non-trivial $p \in P_m(K_n)$ such that

$$\sigma(-1)^k p(t) \geqslant 0, \qquad t \in [t_k^*, t_{k+1}^*], \qquad k=1, ..., r. \tag{12}$$

Since $I_k \subseteq [t_k^*, t_{k+1}^*]$, $k=1, ..., r$, we get by (10), (11), and (12),

$$(f-p_f)(t)\, p(t) \geqslant 0, \qquad t \in E_{f-p_f, [a,b]}$$

which contradicts, by Theorem 7, the strong unicity of p_f.

Now let n be even. Set $K_{n-1} = K_n \setminus \{x_{n-1}\}$. Therefore, $P_m(K_{n-1}) \subseteq P_m(K_n)$. By the same proof as above, it follows that

$$\mathscr{A}(f-p_f)|_{[a,b]} \geqslant n.$$

Let $t_1 < \cdots < t_n$ be $\mathscr{A}$-points of $f-p_f$ in $[a, b)$. We claim that

$$card(E_{f-p_f, [a,b)} \setminus \{t_1, ..., t_n\}) \geqslant 1. \tag{13}$$

To the contrary, suppose that $E_{f-p_f, [a,b)} = \{t_1, ..., t_n\}$. We first assume that the homogeneous interpolation problem (1) with respect to $\{t_1, ..., t_n\}$ has only the trivial solution. Hence, there exists a non-trivial $p \in P_m(K_n)$ such that

$$p(t_k) = (f-p_f)(t_k), \qquad k=1, ..., n.$$

Therefore,

$$(f-p_f)(t)\, p(t) > 0, \qquad t \in E_{f-p_f, [a,b]}. \tag{14}$$

We now assume that there exists a non-trivial $p \in P_m(K_n)$ such that $p(t_k) = 0$, $k = 1, ..., n$. Therefore,

$$(f - p_f)(t)\, p(t) = 0, \qquad t \in E_{f - p_f, [a, b]}. \tag{15}$$

By Theorem 7, (14) (respectively (15)) contradicts the strong unicity of p_f. This shows (13). Thus we get an "additional" extremal point in $[a, b)$. It is now easily seen that (8) holds for a suitable $\xi \in [a, b)$.

We now show that (i) holds. Let $n > m + 1$. Suppose that there exist $j \in \{1, ..., n - m - 1\}$, $i \in \mathbb{Z}$, such that

$$\mathscr{A}(f - p_f)|_{(x_i, x_{i+m+j})} = r \leqslant j. \tag{16}$$

We choose an $\mathscr{A}$-set $\{t_1, ..., t_r\}$ of $f - p_f$ in (x_i, x_{i+m+j}). Thus,

$$\sigma(-1)^k (f - p_f)(t_k) = \|f - p_f\|_\infty, \qquad k = 1, ..., r$$

where $\sigma \in \{-1, 1\}$. By definition of the accompanying alternation intervals $I_k = [\alpha_k, \beta_k]$, $k = 1, ..., r$, of $f - p_f$ in (x_i, x_{i+m+j}), we have

$$\sigma(-1)^k (f - p_f)(t) = \|f - p_f\|_\infty, \qquad t \in E_{f - p_f, (x_i, x_{i+m+j})} \cap I_k, \quad k = 1, ..., r \tag{17}$$

and by (16),

$$E_{f - p_f, (x_i, x_{i+m+j})} \subseteq \bigcup_{k=1}^{r} I_k. \tag{18}$$

Set $t_1^* = x_i$, $t_{r+1}^* = x_{i+m+j}$ and choose $t_k^* \in (\beta_{k-1}, \alpha_k)$, $k = 2, ..., r$. Since $r \leqslant j$, by Lemma 8(i), there exists a non-trivial $s \in S = span\{B_i, ..., B_{i+j-1}\}$ such that

$$\sigma(-1)^k s(t) \geqslant 0, \qquad t \in [t_k^*, t_{k+1}^*], \quad k = 1, ..., r. \tag{19}$$

Since $I_k \subseteq [t_k^*, t_{k+1}^*]$, $k = 1, ..., r$, we get by (17), (18), and (19),

$$(f - p_f)(t)\, s(t) \geqslant 0, \qquad t \in E_{f - p_f, (x_i, x_{i+m+j})}. \tag{20}$$

Let

$$p(t) = \begin{cases} s(t), & \text{if} \quad t \in [x_i, x_{i+m+j}) \\ 0, & \text{if} \quad t \in [x_{i+m+j}, x_{i+n}]. \end{cases}$$

By (20), it is obvious that

$$(f - p_f)(t)\, p(t) \geqslant 0, \qquad t \in E_{f - p_f, [x_i, x_{i+n}]}.$$

Hence, extending p $(b-a)$-periodically on the real line gives a non-trivial spline from $P_m(K_n)$, which contradicts, by Theorem 7, the strong unicity of p_f. This proves Lemma 9. ∎

We remark that the proof of Lemma 9 shows that in the case of odd dimension (8) holds true for $\xi = a$.

3. INTERPOLATION BY PERIODIC SPLINES

In this section we state our results on interpolation by periodic splines. These results will be needed for proving Theorem 3 and Theorem 5. We begin with the following definition.

DEFINITION 10. Let $f \in C$ be a sufficiently differentiable function and a set $\{t_1, ..., t_n\}$ such that $t_1 \leqslant \cdots \leqslant t_n < t_1 + (b-a)$ be given. Set $d_k = \max\{j: t_{k-j} = \cdots = t_k\}$, $k = 1, ..., n$, and assume that $d_k \leqslant m-1$, if $t_k \in \{x_i: i \in \mathbb{Z}\}$ and $d_k \leqslant m$, otherwise. The *Hermite interpolation problem* is to determine a spline $p \in P_m(K_n)$ such that

$$p^{(d_k)}(t_k) = f^{(d_k)}(t_k), \qquad k = 1, ..., n. \tag{21}$$

(Choosing $t_1 < \cdots < t_n$ gives Lagrange interpolation problem (1).) We call $\{t_1, ..., t_n\}$ a *Hermite interpolation set* for $P_m(K_n)$ if for every sufficiently differentiable function $f \in C$ the Hermite interpolation problem (21) has a unique solution from $P_m(K_n)$. If $t_1 < \cdots < t_n$, we call such a set $\{t_1, ..., t_n\}$ an interpolation set for $P_m(K_n)$. For each $g \in C(\mathbb{R})$ having a finite number of zeros in the (non-empty) interval $I \subseteq \mathbb{R}$, we denote by $N_I(g)$ the number of zeros of g (counting multiplicities) in I.

In the case of odd dimension of $P_m(K_n)$ the next lemma on Hermite interpolation was proved by Schumaker [19, Theorem 8.8]. Davydov [2] considered the case of Lagrange interpolation for even dimension. The following result is a characterization of Hermite interpolation sets for $P_m(K_n)$.

LEMMA 11. *Let $\{t_1, ..., t_n\}$ be as in Definition 10 and set $T = \{t_k: k \in \mathbb{Z}\}$, where $t_{i+jn} = t_i + j(b-a)$, $i = 1, ..., n$, $j \in \mathbb{Z} \setminus \{0\}$. Consider the following statements.*

(i) *The set $\{t_1, ..., t_n\}$ is a Hermite interpolation set for $P_m(K_n)$.*

(ii) *For every interval* (x_i, x_{i+m+j}), $j = 1, ..., n-m-1$, $i \in \mathbb{Z}$, *we have*

$$\mathit{card}((x_i, x_{i+m+j}) \cap T) = \mathit{card}(\{k : t_k \in (x_i, x_{i+m+j})\}) \geq j \tag{22}$$

if $n > m+1$.

(iii) *There does not exist a spline* $p \in P_m(K_n)$ *having exactly the set of zeros* $\{t_1, ..., t_n\}$ *in* $[t_1, t_1 + (b-a))$ (*counting multiplicities*).

If n is odd, then (i) *and* (ii) *are equivalent. If n is even, then* (i) *holds if and only if* (ii) *and* (iii) *are satisfied.*

Proof. We first show that (i) implies (ii) in each case. Let $n > m+1$. Suppose, contrary to (ii), that there exist $j \in \{1, ..., n-m-1\}$, $i \in \mathbb{Z}$, such that

$$\mathit{card}((x_i, x_{i+m+j}) \cap T) \leq j-1.$$

This gives a non-trivial spline $s \in S = \mathit{span}\{B_i, ..., B_{i+j-1}\}$, such that

$$s^{(d_k)}(t_k) = 0, \qquad t_k \in (x_i, x_{i+m+j}) \cap T$$

(cf. Nürnberger [16, Theorem 3.7]). Extending p, defined by

$$p(t) = \begin{cases} s(t), & \text{if} \quad t \in (x_i, x_{i+m+j}) \\ 0, & \text{if} \quad t \in [x_{i+m+j}, x_{i+n}], \end{cases}$$

$(b-a)$-periodically on the real line gives a non-trivial solution of the homogeneous problem (21), which contradicts (i). (i) $\Rightarrow$ (iii) is evident.

We next prove that (ii) implies (i) if n is odd, and that (ii) and (iii) imply (i) if n is even. Conversely, suppose that there exists a non-trivial solution $p \in P_m(K_n)$ of the homogeneous problem (21) with respect to $\{t_1, ..., t_n\}$. Assume that p has infinitely many zeros in $[a, b)$. Therefore, $n > m+1$. Choose $j \in \{1, ..., n-m-1\}$, $i \in \mathbb{Z}$, such that p has a finite number of zeros (counting multiplicities) in (x_i, x_{i+m+j}) and vanishes on the left and on the right, i.e.,

$$p(t) = 0, \qquad t \in [x_{i-1}, x_i] \cup [x_{i+m+j}, x_{i+m+j+1}].$$

Since (ii), it follows for $s = p|_{(x_i, x_{i+m+j})} \in S$, $N_{(x_i, x_{i+m+j})}(s) \geq j$, which is a contradiction (cf. Nürnberger [16, Theorem 3.3]). Therefore, the number of zeros of p in $[a, b)$ (counting multiplicities) is finite. By the choice of p, $N_{[a, b)}(p) \geq n$. If n is odd, then this is a contradiction. If n is even, then it follows that $N_{[a, b)}(p) = n$. By the choice of p, this contradicts (iii). This completes the proof of Lemma 11. ∎

Remark 12. Let us mention that (22) holds for each set $\{t_1, ..., t_n\}$ as in Definition 10, if $j \in \{0, n-m\}$. Therefore, we always have (22) if $n \in \{m, m+1\}$. Moreover, it is easy to see that if $n > m+1$, (22) holds for all $j \in \{1, ..., n-m-1\}$, $i \in \mathbb{Z}$, if and only if

$$card([x_i, x_{i+k}] \cap T) \leqslant m+k, \qquad k=1, ..., n-m-1, \quad i \in \mathbb{Z}.$$

(Here, $T = \{t_k : k \in \mathbb{Z}\}$ is defined as in Lemma 11.)

In the beginning of Section 1, we called a set $\{t_1, ..., t_n\}$ such that $t_1 < \cdots < t_n (< t_1 + (b-a))$, for which the contrary of statement (iii) from Lemma 11 holds, a NI-set for $P_m(K_n)$ (see Definition 1.).

LEMMA 13. *Let n be even, $n \geqslant m$, and $\{t_1, ..., t_n\}$ as in Definition* 10 *and set $T = \{t_k : k \in \mathbb{Z}\}$, where $t_{i+jn} = t_i + j(b-a)$, $i = 1, ..., n$, $j \in \mathbb{Z} \backslash \{0\}$. If there exists a spline $p \in P_m(K_n)$ having exactly the set of zeros $\{t_1, ..., t_n\}$ in $[t_1, t_1 + (b-a))$ (counting multiplicities), then for every interval (x_i, x_{i+m+j}), $j = 0, ..., n-m$, $i \in \mathbb{Z}$, we have*

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j+1.$$

Proof. To the contrary, assume that there exist $j \in \{0, ..., n-m\}$, $i \in \mathbb{Z}$, such that

$$card((x_i, x_{i+m+j}) \cap T) \leqslant j.$$

Therefore,

$$card([x_{i+m+j}, x_{i+n}] \cap T) \geqslant n-j.$$

By assumption,

$$s = p|_{[x_{i+m+j}, x_{i+n}]} \in S_m(x_{i+m+j+1}, ..., x_{i+n-1}) = span\{B_{i+j}, ..., B_{i+n-1}\}$$

has a finite number of zeros in $[x_{i+m+j}, x_{i+n}]$. Thus, $N_{[x_{i+m+j}, x_{i+n}]}(s) \geqslant n-j$, which is a contradiction (cf. Nürnberger [16, Theorem 3.3]). The lemma is proved. ∎

By Lemma 11 and Lemma 13, we obtain the following result.

COROLLARY 14. *Let n be even, $n \geqslant m$, and $\{t_1, ..., t_n\}$ as in Definition* 10 *and set $T = \{t_k : k \in \mathbb{Z}\}$, where $t_{i+jn} = t_i + j(b-a)$, $i = 1, ..., n$, $j \in \mathbb{Z} \backslash \{0\}$. Suppose that for every interval (x_i, x_{i+m+j}), $j = 0, ..., n-m$, $i \in \mathbb{Z}$, we have*

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j$$

and there exist $j_0 \in \{0, ..., n-m\}$, $i_0 \in \mathbb{Z}$, *such that*

$$card((x_{i_0}, x_{i_0+m+j_0}) \cap T) = j_0. \tag{23}$$

Then $\{t_1, ..., t_n\}$ *is a Hermite interpolation set for* $P_m(K_n)$.

The next theorem is a generalization of Davydov's results on Lagrange interpolation for even-dimensional $P_m(K_n)$ (cf. Davydov [2, 3], see also Korneichuk [8, Theorem 2.4.9]). Roughly speaking, the following theorem says that each set "between" a NI-set is an interpolation set for $P_m(K_n)$.

THEOREM 15. *Let* n *be even. Suppose that* $\{t_1, ..., t_n\}$ *is a NI-set for* $P_m(K_n)$. *Then every set* $\{t_1^*, ..., t_n^*\}$ *such that* $t_1^* < \cdots < t_n^* < t_1 + (b-a) = t_{n+1}$, *differing from* $\{t_1, ..., t_n\}$, *with*

$$t_k \leqslant t_k^* \leqslant t_{k+1}, \qquad k = 1, ..., n \tag{24}$$

is an interpolation set for $P_m(K_n)$.

Proof. We begin by considering $n \geqslant m$. By Lemma 13, the definition of a NI-set and (24), it follows that

$$card((x_i, x_{i+m+j}) \cap T^*) \geqslant j, \qquad j = 0, ..., n-m, \qquad i \in \mathbb{Z}.$$

(Here, $T^* = \{t_k^* : k \in \mathbb{Z}\}$, where $t_{i+jn}^* = t_i^* + j(b-a)$, $i = 1, ..., n$, $j \in \mathbb{Z} \setminus \{0\}$.) If there exist $j_0 \in \{0, ..., n-m\}$, $i_0 \in \mathbb{Z}$, such that (23) holds for T^* in $(x_{i_0}, x_{i_0+m+j_0})$, then, by Corollary 14, $\{t_1^*, ..., t_n^*\}$ is an interpolation set for $P_m(K_n)$. Therefore, we have to consider the case that

$$card((x_i, x_{i+m+j}) \cap T^*) \geqslant j+1, \qquad j = 0, ..., n-m, \quad i \in \mathbb{Z} \tag{25}$$

or, alternatively, $n < m$.

If $n > m+2$, then (25) implies for each set $T_k^* = T^* \setminus \{t_{k+ln}^* : l \in \mathbb{Z}\}$, $k = 1, ..., n$,

$$card((x_i^{(1)}, x_{i+m+j}^{(1)}) \cap T_k^*) \geqslant j, \qquad j = 1, ..., n-m-2, \quad i \in \mathbb{Z}$$

where $K_{n-1} = K_n \setminus \{x_{n-1}\} = \{x_0^{(1)}, ..., x_{n-1}^{(1)}\}$. By Lemma 11 and the weak Chebyshev property of $P_m(K_{n-1})$, it follows that

$$D_k = D\begin{pmatrix} p_1 & \cdots & p_{k-1} & p_k & \cdots & p_{n-1} \\ t_1^* & \cdots & t_{k-1}^* & t_{k+1}^* & \cdots & t_n^* \end{pmatrix} > 0, \qquad k = 1, ..., n \tag{26}$$

where $\{p_1, ..., p_{n-1}\}$ is a suitable basis of $P_m(K_{n-1})$. Obviously (26) also holds if $n \leqslant m+2$. Now let $p_n \in P_m(K_n)$ be a periodic spline having exactly the set of zeros $\{t_1, ..., t_n\}$ in $[t_1, t_1+(b-a))$. Since $P_m(K_{n-1}) \subseteq P_m(K_n)$, it is easily seen that $\{p_1, ..., p_{n-1}, p_n\}$ is a basis of $P_m(K_n)$ (cf. [23, Satz 1.1.6]). Choose $\sigma \in \{-1, 1\}$ such that

$$\sigma(-1)^k p_n(t) > 0, \qquad t \in (t_k, t_{k+1}), \quad k = 1, ..., n \tag{27}$$

and $\gamma = \{k \in \{1, ..., n\} : t_k < t_k^* < t_{k+1}\}$. By assumption, $\gamma \neq \emptyset$. Laplace expansion now yields

$$D = D\begin{pmatrix} p_1 & \cdots & p_n \\ t_1^* & \cdots & t_n^* \end{pmatrix} = \sum_{k \in \gamma} (-1)^{n+k} p_n(t_k^*)\, D_k.$$

By (24), (26), and (27), it follows that

$$\sigma(-1)^k p_n(t_k^*)\, D_k > 0, \qquad k \in \gamma.$$

Thus, $D \neq 0$, which proves Theorem 15. ▮

Remark 16. The result of Theorem 15 has consequences for the theory of interpolation by periodic spline spaces of even dimension. It follows that in each neighbourhood of a NI-set for $P_m(K_n)$, one can find interpolation sets for $P_m(K_n)$. In particular, if a NI-set $\{t_1, ..., t_n\}$ for $P_m(K_n)$ is given, then every set $\{t_1, ..., t_{k-1}, t_{k+1}, ..., t_n\} \cup \{t^*\}$ with $t^* \in [t_1, t_1+(b-a))$ and $t^* \notin \{t_1, ..., t_n\}$ is an interpolation set for $P_m(K_n)$.

Moreover, since $\{x_0, ..., x_{n-1}\}$ is a NI-set for $P_{2l}(K_n)$, $l \in \mathbb{N}$ (cf. Krinzessa [7, Satz 12]), we conclude from Theorem 15 that each set $\{t_1^*, ..., t_n^*\}$ such that $t_1^* < \cdots < t_n^* < x_n$, differing from $\{x_0, ..., x_{n-1}\}$ with $x_{k-1} \leqslant t_k^* \leqslant x_k$, $k = 1, ..., n$, is an interpolation set for $P_{2l}(K_n)$. For the case of odd degree see [23, 2.2.5–2.2.6].

Remark 17. It can be seen easily (cf. [23, Satz 2.1.3]) that the converse of Theorem 15 holds in the following sense. For every interpolation set $\{t_1^*, ..., t_n^*\}$ for $P_m(K_n)$, there exists a NI-set $\{t_1, ..., t_n\}$ for $P_m(K_n)$ such that $t_k^* < t_k < t_{k+1}^*$, $k = 1, ..., n$.

4. PROOF OF THEOREM 3

In this section we prove Theorem 3. We show that the necessary conditions given in Lemma 9 are also sufficient if $P_m(K_n)$ is of odd dimension. For proving this, we need the following lemma which is a consequence of the weak Chebyshev property of $S_m(x_1, ..., x_{n-1}) = span\{B_{-m}, ..., B_{n-1}\}$ (cf. Nürnberger [16, Lemma 1.11]).

LEMMA 18. *Let points $a \leqslant t_1 < \cdots < t_{n+m+1} \leqslant b$ be given. The following statements are equivalent.*

(i) *There does not exist a non-trivial spline $s \in S_m(x_1, ..., x_{n-1})$ such that*

$$(-1)^k s(t_k) \geqslant 0, \qquad k = 1, ..., n+m+1.$$

(ii) *For all $k \in \{1, ..., n+m+1\}$,*

$$D\begin{pmatrix} B_{-m} & & \cdots & \cdots & & B_{n-1} \\ t_1 & \cdots & t_{k-1} & t_{k+1} & \cdots & t_{n+m+1} \end{pmatrix} \neq 0.$$

In the following we show that if n is odd, then statement (ii) of Theorem 3 implies the strong unicity of $p_f \in P_m(K_n)$. Together with Lemma 9 this proves Theorem 3.

Proof of Theorem 3, (ii) $\Rightarrow$ (i). Let us first consider the case $n \leqslant m+1$, i.e., $P_m(K_n)$ is a Chebyshev space. Since (3), by the alternation theorem (cf. Nürnberger [16, Theorem 3.12]), p_f is a best uniform approximation. Moreover, p_f is unique. By McLaughlin and Somers [9] unique and strongly unique best uniform approximations from Chebyshev spaces coincide. This finishes the proof for $n \leqslant m+1$.

We now turn to the case $n > m+1$. Suppose that p_f is not a strongly unique best uniform approximation of f from $P_m(K_n)$. By Theorem 7 there exists a non-trivial $p \in P_m(K_n)$ such that

$$(f - p_f)(t)\, p(t) \geqslant 0, \qquad t \in E_{f-p_f, [a,b]}. \tag{28}$$

Let us first assume that p has infinitely many zeros in $[a, b)$. Choose $j \in \{1, ..., n-m-1\}$ and $i \in \mathbb{Z}$ such that p has a finite number of zeros (counting multiplicities) in (x_i, x_{i+m+j}) and vanishes on the left and on the right, i.e.,

$$p(t) = 0, \qquad t \in [x_{i-1}, x_i] \cup [x_{i+m+j}, x_{i+m+j+1}].$$

By (2), it follows that

$$\mathscr{A}(f - p_f)|_{[x_i, x_{i+m+j}]} \geqslant j+1.$$

Set $S = span\{B_i, ..., B_{i+j-1}\}$. Going to subintervals, if necessary, it follows that there exists an interval $[x_r, x_{r+q}]$, $q \geqslant 1$, such that $f - p_f$ has $d+1$ $\mathscr{A}$-points $t_1 < \cdots < t_{d+1}$ in $[x_r, x_{r+q}]$, where $d = dim S|_{[x_r, x_{r+q}]}$, but every proper subinterval $[x_{r^*}, x_{r^*+q^*}]$, $q^* \geqslant 1$, of $[x_r, x_{r+q}]$ contains at most d^* points from $\{t_1, ..., t_{d+1}\}$, where $d^* = dim S|_{[x_{r^*}, x_{r^*+q^*}]}$. Choose $l_1, l_2 \in \{0, ..., m\}$ such that

$$S|_{[x_r, x_{r+q}]} = span\{B_{r-l_1}, ..., B_{r+q-1-l_2}\}.$$

Therefore, $q + l_1 - l_2 = d$. We next claim that for all $k \in \{1, ..., d+1\}$,

$$D\begin{pmatrix} B_{r-l_1} & & \cdots & \cdots & & B_{r+q-1-l_2} \\ t_1 & \cdots & t_{k-1} & t_{k+1} & \cdots & t_{d+1} \end{pmatrix} \neq 0. \tag{29}$$

Conversely, suppose that there exists $k \in \{1, ..., d+1\}$ such that

$$card((x_{i_1}, x_{i_1+m+j_1}) \cap \{t_1, ..., t_{k-1}, t_{k+1}, ..., t_{d+1}\}) \leqslant j_1 - 1 \tag{30}$$

where $i_1 \in \{r - l_1, ..., r + q - 1 - l_2\}$, $j_1 \in \{1, ..., r + q - l_2 - i_1\}$. Obviously, $[x_r, x_{r+q}] \backslash (x_{i_1}, x_{i_1+m+j_1}) \neq \emptyset$. We consider five cases.

Case 1. Let $i_1 > r$ and $i_1 + m + j_1 < r + q$.
By the choice of $\{t_1, ..., t_{d+1}\}$, we have

$$card([x_r, x_{i_1}] \cap \{t_1, ..., t_{d+1}\}) \leqslant i_1 - r + l_1.$$

Therefore, it follows from (30),

$$card([x_{i_1+m+j_1}, x_{r+q}] \cap \{t_1, ..., t_{d+1}\}) \geqslant d + 1 + r - i_1 - j_1 - l_1$$

Since $dim\ S|_{[x_{i_1+m+j_1}, x_{r+q}]} = d + r - i_1 - j_1 - l_1$, this contradicts the choice of $[x_r, x_{r+q}]$.

Case 2. Let $i_1 > r$ and $i_1 + m + j_1 > r + q$, or $i_1 < r$ and $i_1 + m + j_1 < r + q$. Since the proof of the second of these two cases is similar, we only consider $i_1 > r$ and $i_1 + m + j_1 > r + q$. By (30),

$$card([x_r, x_{i_1}] \cap \{t_1, ..., t_{d+1}\}) \geqslant d + 1 - j_1.$$

Since $i_1 + j_1 - r \leqslant q - l_2$ and $d = q + l_1 - l_2$, we have $d + 1 - j_1 \geqslant i_1 - r + l_1 + 1$. But $dim\ S|_{[x_r, x_{i_1}]} = i_1 - r + l_1$, which contradicts the choice of $[x_r, x_{r+q}]$.

Case 3. Let $i_1 = r$ and $i_1 + m + j_1 < r + q$, or $i_1 > r$ and $i_1 + m + j_1 = r + q$.
Since the proof of the second of these two cases is similar, we only consider $i_1 = r$ and $m + j_1 < q$. By (30),

$$card([x_{r+m+j_1}, x_{r+q}] \cap \{t_1, ..., t_{d+1}\}) \geqslant d - j_1 + \delta_{0, l_1}$$

where δ_{0, l_1} denotes Kronecker's symbol. But $dim\, S|_{[x_{r+m+j_1}, x_{r+q}]} = d - j_1 - l_1$, which contradicts the choice of $[x_r, x_{r+q}]$.

Case 4. Let $i_1 = r$ and $i_1 + m + j_1 > r + q$, or $i_1 < r$ and $i_1 + m + j_1 = r + q$.

Since the proof of the second of these two cases is similar, we only consider $i_1 = r$ and $m + j_1 > q$. Since $j_1 \leqslant q - l_2$, we have by (30),

$$card([x_r, x_{r+q}] \cap \{t_1, ..., t_{d+1}\}) \leqslant q - l_2 + 1 - \delta_{0, l_1} < d + 1$$

which contradicts the choice of $[x_r, x_{r+q}]$.

Case 5. Let $i_1 = r$ and $i_1 + m + j_1 = r + q$, i.e., $j_1 = q - m$.

By (30),

$$card([x_r, x_{r+q}] \cap \{t_1, ..., t_{d+1}\}) \leqslant q - m + 2 - \delta_{0, l_1} - \delta_{m, l_2}.$$

Therefore,

$$card([x_r, x_{r+q}] \cap \{t_1, ..., t_{d+1}\}) < d + 1$$

which contradicts the choice of $[x_r, x_{r+q}]$.

Thus, (29) holds for all $k \in \{1, ..., d+1\}$. Since $\{t_1, ..., t_{d+1}\}$ is an $\mathscr{A}$-set of $f - p_f$ in $[x_r, x_{r+q}]$, it follows that there exists $\sigma \in \{-1, 1\}$ such that

$$\sigma(-1)^k (f - p_f)(t_k) = \|f - p_f\|_\infty, \qquad k = 1, ..., d+1. \tag{31}$$

Set $s = p|_{[x_r, x_{r+q}]} \in S|_{[x_r, x_{r+q}]}$. By (28) and (31),

$$\sigma(-1)^k s(t_k) \geqslant 0, \qquad k = 1, ..., d+1.$$

Lemma 18 now implies that $s \equiv 0$ which contradicts the choice of (x_i, x_{i+m+j}). Consequently, p has only a finite number of zeros in $[a, b)$. Since n is odd, $N_{[a, b)}(p) \leqslant n - 1$. Therefore, the cardinality of each set $\{t_1^*, ..., t_r^*\}$ with $t_1^* < \cdots < t_r^* < t_1^* + (b - a)$ such that

$$\sigma^*(-1)^k p(t_k^*) \geqslant 0, \qquad k = 1, ..., r$$

where $\sigma^* \in \{-1, 1\}$, is at most n. Since (3), there exists an $\mathscr{A}$-set $\{t_1, ..., t_{n+1}\}$ of $f - p_f$ in $[a, b]$, i.e.,

$$\sigma(-1)^k (f - p_f)(t_k) = \|f - p_f\|_\infty, \qquad k = 1, ..., n+1$$

where $\sigma \in \{-1, 1\}$. By (28), it follows that

$$\sigma(-1)^k p(t_k) \geqslant 0, \qquad k = 1, ..., n+1$$

which is a contradiction. This completes the proof of Theorem 3. ∎

At the end of this section we give a result on strong unicity, which treats the case of exactly $n+1$ extremal points of the error.

COROLLARY 19. *Let n be odd, $f \in C \backslash P_m(K_n)$ and a spline $p_f \in P_m(K_n)$ be given. Suppose that*

$$card(E_{f-p_f, [a, b)}) = n+1.$$

Then the following statements are equivalent.

(i) *The spline p_f is a strongly unique best uniform approximation of f from $P_m(K_n)$.*

(ii) *There exists an $\mathscr{A}$-set $\{t_1, ..., t_{n+1}\}$ of $f-p_f$ in $[a, b]$ such that each set $\{t_1, ..., t_{n+1}\} \backslash \{t_k\}$, $k=1, ..., n+1$, is an interpolation set for $P_m(K_n)$.*

Proof. The case $n \leqslant m+1$ is obvious. Let us consider $n > m+1$. We first show that (i) $\Rightarrow$ (ii). By Theorem 3,

$$\mathscr{A}(f-p_f)|_{[a, b]} \geqslant n+1.$$

Therefore, the points $t_1 < \cdots < t_{n+1}$ such that $E_{f-p_f, [a, b)} = \{t_1, ..., t_{n+1}\}$ are the $\mathscr{A}$-points of $f-p_f$ in $[a, b]$. Set $T = \{t_{k+ln} = t_k + l(b-a)$: $k=1, ..., n+1,\ l \in \mathbb{Z}\}$. By Theorem 3,

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j+1, \qquad j=1, ..., n-m-1, \quad i \in \mathbb{Z}.$$

Hence, by Lemma 11, it follows (ii).

We now show that (ii) $\Rightarrow$ (i). Let us assume that p_f is not a strongly unique best uniform approximation of f from $P_m(K_n)$. Since $\{t_1, ..., t_{n+1}\}$ is an $\mathscr{A}$-set of $f-p_f$ in $[a, b]$, it follows from Theorem 3, that there exist $j \in \{1, ..., n-m-1\}$, $i \in \mathbb{Z}$, such that

$$card((x_i, x_{i+m+j}) \cap T) \leqslant j.$$

Here T is given as above. Therefore, there exists $k \in \{1, ..., n+1\}$ such that

$$card((x_i, x_{i+m+j}) \cap T_k) \leqslant j-1$$

where $T_k = T \backslash \{t_{k+ln}: l \in \mathbb{Z}\}$. By Lemma 11, this contradicts (ii). The corollary is proved. ∎

5. PROOF OF THEOREM 5

In this section we prove Theorem 5 which gives a complete characterization of the strongly unique best uniform approximation from $P_m(K_n)$ in the case of even dimension. For doing this, we need the interpolation results of Section 3 and the following lemmas.

LEMMA 20. *Let $n > m+1$ and $\alpha_k \leqslant \beta_k < \alpha_{k+1}$, $k = 1, ..., n$, with $\alpha_1 + (b-a) = \alpha_{n+1}$ be given. Let $I_k = [\alpha_k, \beta_k]$, $k = 1, ..., n$, and set*

$$I_{k+ln} = [\alpha_k + l(b-a), \beta_k + l(b-a)], \qquad k = 1, ..., n, \quad l \in \mathbb{Z} \backslash \{0\}.$$

Suppose that for every interval (x_i, x_{i+m+j}), $j = 1, ..., n-m$, $i \in \mathbb{Z}$,

$$card(\{I_l \colon I_l \cap (x_i, x_{i+m+j}) \neq \emptyset, l \in \mathbb{Z}\}) \geqslant j+1. \tag{32}$$

Then there exists $\{t_1^*, ..., t_n^*\}$ with $t_k^* \in I_k$, $k = 1, ..., n$, and $t_k^* \in I_k^0$ whenever I_k^0 is non-empty, such that for each set $\{t_1, ..., t_n\}$ with $\alpha_k \leqslant t_k \leqslant t_k^*$, $k = 1, ..., n$, or $t_k^* \leqslant t_k \leqslant \beta_k$, $k = 1, ..., n$,

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j, \qquad j = 1, ..., n-m-1, \quad i \in \mathbb{Z}$$

where $T = \{t_{k+ln} = t_k + l(b-a) \colon k = 1, ..., n, l \in \mathbb{Z}\}$.

Proof. We first describe how to choose $\{t_1^*, ..., t_n^*\}$. Let $k \in \{1, ..., n\}$. Set $t_k^* = (\alpha_k + \beta_k)/2$, if $x_i \notin I_k^0$, $i \in \mathbb{Z}$. Otherwise, choose $i_1 \in \mathbb{Z}$ and $j_1 \in \{0, ..., m\}$ such that

$$\{x_{i_1}, ..., x_{i_1+j_1}\} = I_k^0 \cap \{x_i \colon i \in \mathbb{Z}\}$$

and consider the following cases.

Case 1. For all $j = 1, ..., n-m-1$, and $i \in \{i_1, ..., i_1+j_1\}$,

$$card(\{I_l \colon I_l \subseteq (x_i, x_{i+m+j}), l \in \mathbb{Z}\}) \geqslant j \tag{33}$$

and

$$card(\{I_l \colon I_l \subseteq (x_{i-m-j}, x_i), l \in \mathbb{Z}\}) \geqslant j. \tag{34}$$

Choose $t_k^* \in I_k^0$.

Case 2. There exist $j^* \in \{1, ..., n-m-1\}$ and $i^* \in \{i_1, ..., i_1+j_1\}$ such that

$$card(\{I_l \colon I_l \subseteq (x_{i^*}, x_{i^*+m+j^*}), l \in \mathbb{Z}\}) = j^* - 1 \tag{35}$$

and (34) holds for all $j=1, ..., n-m-1$, and $i \in \{i_1, ..., i_1+j_1\}$. Choose $i_0^* \in \{i_1, ..., i_1+j_1\}$ maximal such that (35) holds for a suitable $j_0^* \in \{1, ..., n-m-1\}$. If $i_0^* < i_1+j_1$, then choose $t_k^* \in (x_{i_0^*}, x_{i_0^*+1})$, else choose $t_k^* \in (x_{i_1+j_1}, \beta_k)$.

Case 3. There exist $j^{**} \in \{1, ..., n-m-1\}$ and $i^{**} \in \{i_1, ..., i_1+j_1\}$ such that

$$card(\{I_l\colon I_l \subseteq (x_{i^{**}-m-j^{**}}, x_{i^{**}}), l \in \mathbb{Z}\}) = j^{**}-1 \tag{36}$$

and (33) holds for all $j=1, ..., n-m-1$, and $i \in \{i_1, ..., i_1+j_1\}$. Choose $i_0^{**} \in \{i_1, ..., i_1+j_1\}$ minimal such that (36) holds for a suitable $j_0^{**} \in \{1, ..., n-m-1\}$. If $i_0^{**} > i_1$, then choose $t_k^* \in (x_{i_0^{**}-1}, x_{i_0^{**}})$, else choose $t_k^* \in (\alpha_k, x_{i_1})$.

Case 4. There exist $j^* \in \{1, ..., n-m-1\}$ and $i^* \in \{i_1, ..., i_1+j_1\}$ such that (35) holds and there exist $j^{**} \in \{1, ..., n-m-1\}$ and $i^{**} \in \{i_1, ..., i_1+j_1\}$ such that (36) holds. Choose i_0^*, j_0^* and i_0^{**}, j_0^{**} as in Case 2, respectively Case 3. Therefore, we obtain by (32),

$$card(\{I_l\colon I_l \cap (x_{i_0^{**}-m-j_0^{**}}, x_{i_0^{**}}) \neq \emptyset, l \in \mathbb{Z}\}) = j_0^{**}+1$$

and

$$card(\{I_l\colon I_l \cap (x_{i_0^*}, x_{i_0^*+m+j_0^*}) \neq \emptyset, l \in \mathbb{Z}\}) = j_0^*+1.$$

Since $\{x_{i_1}, ..., x_{i_1+j_1}\} \subseteq I_k^0$,

$$\gamma = card(\{I_l\colon I_l \cap (x_{i_0^{**}-m-j_0^{**}}, x_{i_0^*+m+j_0^*}) \neq \emptyset, l \in \mathbb{Z}\}) = j_0^* + j_0^{**}+1.$$

We first consider the case $i_0^* - i_0^{**} + m + j_0^* + j_0^{**} \leqslant n-1$. By (32), $\gamma \geqslant i_0^* - i_0^{**} + m + j_0^* + j_0^{**} + 1$ ((32) even holds for $j = n-m+1, ..., n-1$, since for these choices of j, $card(\{I_l\colon I_l \cap (x_i, x_{i+m+j}) \neq \emptyset, l \in \mathbb{Z}\}) \geqslant card(\{I_l\colon I_l \cap (x_i, x_{i+n}] \neq \emptyset, l \in \mathbb{Z}\}) \geqslant n \geqslant j+1$). Thus, in this case, $i_0^{**} \geqslant i_0^* + m$.

Now let $i_0^* - i_0^{**} + m + j_0^* + j_0^{**} \geqslant n$. By (32),

$$\begin{aligned} &card(\{I_l\colon I_l \subseteq (x_{i_0^{**}+n-j_0^{**}-m}, x_{i_0^*+m+j_0^*}), l \in \mathbb{Z}\}) \\ &\quad \geqslant i_0^* - i_0^{**} + m + j_0^* + j_0^{**} - n - 1. \end{aligned}$$

Since

$$card(\{I_l\colon I_l \cap [x_{i_0^*}, x_{i_0^{**}+n-j_0^{**}-m}] \neq \emptyset, l \in \mathbb{Z}\}) \geqslant n - j_0^{**}+1$$

we have $j_0^* - 1 \geqslant n - j_0^{**} + i_0^* - i_0^{**} + m + j_0^* + j_0^{**} - n - 1$. Thus, $i_0^{**} \geqslant i_0^* + m$.

Hence, in both cases, it follows that $i_0^* = i_1$, $i_0^{**} = i_1 + j_1$ and $j_1 = m$. Choose $t_k^* \in (x_{i_0^*}, x_{i_0^{**}}) = (x_{i_1}, x_{i_1+m})$.

We now prove that $\{t_1^*, ..., t_n^*\}$ has the desired property. To the contrary, assume that there exist $\{t_1, ..., t_n\}$ with $\alpha_k \leqslant t_k \leqslant t_k^*$, $k = 1, ..., n$, and $j \in \{1, ..., n-m-1\}$, $i \in \mathbb{Z}$, such that

$$card((x_i, x_{i+m+j}) \cap T) \leqslant j-1.$$

Therefore, by (32) and $t_l \in I_l$, $l \in \mathbb{Z}$, it follows that

$$\{I_l \colon I_l \subseteq (x_i, x_{i+m+j}), l \in \mathbb{Z}\} = \{I_{l_1+1}, ..., I_{l_1+j-1}\}$$

and $x_{i+m+j} \in I^0_{l_1+j}$ for a suitable $l_1 \in \mathbb{Z}$. Obviously, $t_{l_1+k} \in I_{l_1+k}$, $k = 1, ..., j-1$. Hence, $t_{l_1+j} \geqslant x_{i+m+j}$. Set $T^* = \{t^*_{k+ln} = t_k^* + l(b-a),\ k = 1, ..., n,\ l \in \mathbb{Z}\}$. By the choice of $t^*_{l_1+j}$ (see Case 3 and Case 4), $\alpha_{l_1+j} \leqslant t_{l_1+j} \leqslant t^*_{l_1+j} < x^{**}_{i_0}$ for the chosen $x^{**}_{i_0} \leqslant x_{i+m+j}$, which is a contradiction. The case $t_k^* \leqslant t_k \leqslant \beta_k$, $k = 1, ..., n$, uses Case 2 and Case 4 and is similar. This completes the proof of Lemma 20. ∎

Lemma 21. *Let n be even, $r \in \mathbb{N}$, $2r-2 < n$, and points $t_1 < \cdots < t_{2r-2} < t_1 + (b-a)$ be given. If $2r > m+2$, suppose that*

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j-1-n+2r, \qquad j = n-2r+1, ..., n-m, \quad i \in \mathbb{Z},$$

where $T = \{t_{k+ln} = t_k + l(b-a),\ k = 1, ..., 2r-2,\ l \in \mathbb{Z}\}$. Then there exists $p \in P_m(K_n)$ having exactly the set of zeros $\{t_1, ..., t_{2r-2}\}$ in $[t_1, t_1 + (b-a))$ and all of these zeros are simple.

Proof. Let us first consider the case $2r-1 \leqslant m+1$, i.e., $P_m(K_{2r-1})$, where $K_{2r-1} = K_n \setminus \{x_{2r-1}, ..., x_{n-1}\}$, is a Chebyshev space. Choose $t_{2r-1} \in (t_{2r-2}, t_1 + (b-a))$. Hence, there exists $p \in P_m(K_{2r-1})$ such that $p(t_{2r-1}) = 1$ and $p(t_k) = 0$, $k = 1, ..., 2r-2$. Since $N_{[a,b)}(p) = 2r-2$, and $P_m(K_{2r-1}) \subseteq P_m(K_n)$, the proof is finished for $2r-1 \leqslant m+1$. We now turn to the case $2r > m+2$. We claim that there exists $K_{n-1} = \{x_0^{(1)}, ..., x_{n-1}^{(1)}\} \subset K_n$ such that

$$card((x_i^{(1)}, x^{(1)}_{i+m+j}) \cap T) \geqslant j-n+2r, \qquad j = n-2r, ..., n-1-m, \quad i \in \mathbb{Z}. \tag{37}$$

If $card((x_i, x_{i+m+j}) \cap T) \geqslant j-n+2r$, $j = n-2r+1, ..., n-m$, $i \in \mathbb{Z}$, then set $K_{n-1} = K_n \setminus \{x_{n-1}\}$. In this case, (37) obviously holds. Now let us choose $j^* \in \{n-2r+1, ..., n-m\}$ minimal such that

$$card((x_{i^*}, x_{i^*+m+j^*}) \cap T) = j^*-1-n+2r$$

holds for a suitable $i^* \in \mathbb{Z}$. Set $K_{n-1} = K_n \setminus \{x_{i^*}\} = \{x_0^{(1)}, ..., x_{n-1}^{(1)}\}$. If $x_{i^*} \in (x_i^{(1)}, x_{i+m+j}^{(1)})$, then

$$card((x_i^{(1)}, x_{i+m+j}^{(1)}) \cap T) = card((x_i, x_{i+1+m+j}) \cap T) \geqslant j - n + 2r.$$

Let us now assume that there exist $j \in \{n-2r+1, ..., n-1-m\}$, $i \in \mathbb{Z}$, with $x_{i^*} \notin (x_i^{(1)}, x_{i+m+j}^{(1)})$ and

$$j - 1 - n + 2r = card((x_i^{(1)}, x_{i+m+j}^{(1)}) \cap T) = card((x_i, x_{i+m+j}) \cap T).$$

We have to consider two cases.

Case 1. Let $i^* < i < i^* + m + j^* \leqslant i + m + j \leqslant i^* + n$.
By the minimal choice of j^*,

$$\begin{aligned} card((x_{i^*}, x_{i^*+m+j^*}) \cap T) &= \overbrace{card((x_{i^*}, x_{i+m+j}) \cap T)}^{\geqslant i - i^* + j - 1 - n + 2r} - \overbrace{card((x_i, x_{i+m+j}) \cap T)}^{= j - 1 - n + 2r} \\ &\quad + \overbrace{card((x_i, x_{i^*+m+j^*}) \cap T)}^{> i^* - i + j^* - 1 - n + 2r} \\ &> j^* - 1 - n + 2r. \end{aligned} \tag{38}$$

This is a contradiction to the choice of j^*. (Note that if $i^* - i + j^* < n - 2r + 1$, then $card((x_i, x_{i^*+m+j^*}) \cap T) \geqslant 0$ gives (38).)

Case 2. Let $i^* < i^* + m + j^* \leqslant i < i + m + j \leqslant i^* + n$.
Since $n > 2r - 2$,

$$\begin{aligned} card((x_{i^*}, x_{i^*+m+j^*}) \cap T) = {} & \overbrace{card((x_{i^*}, x_{i+m+j}) \cap T)}^{\geqslant i - i^* + j - 1 - n + 2r} \\ & - \overbrace{card((x_i, x_{i+m+j}) \cap T)}^{= j - 1 - n + 2r} \\ & - \overbrace{card([x_{i^*+m+j^*}, x_i] \cap T)}^{\leqslant i - i^* - j^* - 1} \\ \geqslant {} & j^* + 1 > j^* - 1 - n + 2r \end{aligned} \tag{39}$$

which contradicts the choice of j^*. (Note that if $i^* - i + j^* + n < n - 2r + 1$, then $card([x_{i^*+m+j^*}, x_i] \cap T) \leqslant 2r - 2$ gives (39).)

We have shown that K_{n-1} has property (37). Proceeding by induction yields $K_{2r-1}=\{x_0^{(n-2r+1)}, ..., x_{2r-1}^{(n-2r+1)}\}\subset K_n$ such that

$$card((x_i^{(n-2r+1)}, x_{i+m+j}^{(n-2r+1)})\cap T)\geqslant j, \qquad j=0, ..., 2r-1-m, \quad i\in\mathbb{Z}. \tag{40}$$

Choose $t_{2r-1}\in(t_{2r-2}, t_1+(b-a))$. From Lemma 11 and (40), it follows that there exists $p\in P_m(K_{2r-1})$ such that $p(t_{2r-1})=1$ and $p(t_k)=0$, $k=1, ..., 2r-2$. Since (40), by a similar argumentation as in the proof of Lemma 11, p has only a finite number of zeros in $[a, b)$. Since $N_{[a,b)}(p)=2r-2$ and $P_m(K_{2r-1})\subseteq P_m(K_n)$, p has the desired properties. This completes the proof of Lemma 21. ∎

We now prove Theorem 5.

Proof of Theorem 5. We first prove that (i) ⇒ (ii). By Lemma 9, (4) and (5) hold. Let $\xi^*\in[a, b)$ be such that $f-p_f$ has exactly n alternation intervals $I_k=[\alpha_k, \beta_k]$, $k=1, ..., n$, in $[\xi^*, \xi^*+(b-a)]$ and choose $\sigma\in\{-1, 1\}$ such that

$$\begin{gathered}\sigma(-1)^k\,(f-p_f)(t)=\|f-p_f\|_\infty,\\ t\in E_{f-p_f,\,[\xi^*,\,\xi^*+(b-a)]}\cap I_k, \qquad k=1, ..., n.\end{gathered}\tag{41}$$

We extend $\{I_k,\ k=1, ..., n\}$ as in the formulation of Lemma 20. If $n>m+1$, then by (4), it follows that

$$\begin{gathered}card(\{I_l: I_l\cap(x_i, x_{i+m+j})\neq\emptyset, l\in\mathbb{Z}\})\geqslant j+1,\\ j=1, ..., n-m-1, \qquad i\in\mathbb{Z}.\end{gathered}\tag{42}$$

We claim that each set $\{\tilde{t}_1, ..., \tilde{t}_n\}$ such that $\tilde{t}_1<\cdots<\tilde{t}_n<\tilde{t}_1+(b-a)=\tilde{t}_{n+1}$ with $\tilde{t}_k\in[\beta_k, \alpha_{k+1}]$, $k=1, ..., n$, is an interpolation set for $P_m(K_n)$. Suppose the contrary. If $n>m+1$, it follows from (42) that

$$card((x_i, x_{i+m+j})\cap\tilde{T})\geqslant j, \qquad j=1, ..., n-m-1, \quad i\in\mathbb{Z}$$

where $\tilde{T}=\{\tilde{t}_{k+ln}=\tilde{t}_k+l(b-a): k=1, ..., n,\ l\in\mathbb{Z}\}$. By Lemma 11, there exists $p\in P_m(K_n)$ having exactly the simple zeros $\{\tilde{t}_1, ..., \tilde{t}_n\}$ in $[\tilde{t}_1, \tilde{t}_1+(b-a))$. Replacing p by $-p$, if necessary, we have

$$\sigma(-1)^{k+1}p(t)\geqslant 0, \qquad t\in[\tilde{t}_k, \tilde{t}_{k+1}], \quad k=1, ..., n. \tag{43}$$

Since $E_{f-p_f,\,[\xi^*,\,\xi^*+(b-a)]}\cap I_{k+1}\subseteq[\tilde{t}_k, \tilde{t}_{k+1}]$, $k=1, ..., n$, we conclude from (41) and (43),

$$(f-p_f)(t)\,p(t)\geqslant 0, \qquad t\in E_{f-p_f,\,[\xi^*,\,\xi^*+(b-a)]}\cap I_k, \quad k=1, ..., n. \tag{44}$$

Since

$$E_{f-p_f,\,[\xi^*,\,\xi^*+(b-a)]}\subseteq\bigcup_{k=1}^{n} I_k$$

it follows from (44) that

$$(f-p_f)(t)\,p(t)\geqslant 0,\qquad t\in E_{f-p_f,\,[\xi^*,\,\xi^*+(b-a)]}.$$

By Theorem 7, this contradicts the strong unicity of p_f.

We next prove that there exists a NI-set $\{t_1, ..., t_n\}$ for $P_m(K_n)$ such that $t_k\in I_k$, $k=1, ..., n$, and $t_k\in I_k^0$ whenever $I_k^0\neq\emptyset$. Set $P_m(K_n)=$ *span*$\{p_1, ..., p_n\}$. Define $D_0\colon[0,1]\mapsto\mathbb{R}$, by

$$D_0(\tau)=D\begin{pmatrix} p_1 & \cdots & p_n\\ \psi_1(\tau) & \cdots & \psi_n(\tau)\end{pmatrix},\qquad \tau\in[0,1],$$

where $\psi_k(\tau)=\beta_k+\tau(\alpha_{k+1}-\beta_k)$, $\tau\in[0,1]$, $k=1, ..., n$. By the above, it follows that $D_0(\tau)\neq 0$, $\tau\in[0,1]$. By continuity of D_0, it follows that

$$D_0(0)\,D_0(1)=D\begin{pmatrix} p_1 & \cdots & p_n\\ \beta_1 & \cdots & \beta_n\end{pmatrix}D\begin{pmatrix} p_1 & \cdots & p_{n-1} & p_n\\ \alpha_2 & \cdots & \alpha_n & \alpha_1\end{pmatrix}>0. \qquad (45)$$

(Note that by (5), $\alpha_{k^*}<\beta_{k^*}$ for at least one $k^*\in\{1, ..., n\}$, and hence $\{\beta_1, ..., \beta_n\}\neq\{\alpha_1, ..., \alpha_n\}$.) Now let $\{t_1^*, ..., t_n^*\}$ be such that $t_k^*\in I_k$, $k=1, ..., n$, and $t_k^*\in I_k^0$ whenever $I_k^0\neq\emptyset$. In addition, since (42), if $n>m+1$, we choose $\{t_1^*, ..., t_n^*\}$ as in Lemma 20. Define $D_1\colon[0,1]\mapsto\mathbb{R}$, by

$$D_1(\tau)=D\begin{pmatrix} p_1 & \cdots & p_n\\ \phi_1(\tau) & \cdots & \phi_n(\tau)\end{pmatrix},\qquad \tau\in[0,1]$$

where

$$\phi_k(\tau)=\begin{cases}\alpha_k+2\tau(t_k^*-\alpha_k), & \text{if}\quad \tau\in[0,\tfrac{1}{2}]\\ \beta_k+2(1-\tau)(t_k^*-\beta_k), & \text{if}\quad \tau\in(\tfrac{1}{2},1]\end{cases}\qquad k=1, ..., n.$$

By (45),

$$D_1(0)\,D_1(1)=D\begin{pmatrix} p_1 & \cdots & p_n\\ \alpha_1 & \cdots & \alpha_n\end{pmatrix}D\begin{pmatrix} p_1 & \cdots & p_n\\ \beta_1 & \cdots & \beta_n\end{pmatrix}=-D_0(0)\,D_0(1)<0.$$

Since (5), by continuity of D_1, there exists $\tau_0 \in (0, 1)$ such that $D_1(\tau_0) = 0$. Set $t_k = \phi_k(\tau_0)$, $k = 1, ..., n$. Clearly, $t_k \in I_k$, $k = 1, ..., n$, and $t_k \in I_k^0$ whenever $I_k^0 \neq \varnothing$. By the choice of $\{t_1^*, ..., t_n^*\}$, if $n > m+1$, it follows from Lemma 20 that

$$card((x_i, x_{i+m+j}) \cap T) \geqslant j, \qquad j = 1, ..., n-m-1, \quad i \in \mathbb{Z}$$

where $T = \{t_{k+ln} = t_k + l(b-a) : k = 1, ..., n, \ l \in \mathbb{Z}\}$. Hence, by Lemma 11, $\{t_1, ..., t_n\}$ is a NI-set for $P_m(K_n)$, which has all the desired properties.

We now show that (ii) $\Rightarrow$ (i). To the contrary, suppose that p_f is not a strongly unique best uniform approximation of f from $P_m(K_n)$. By Theorem 7, there exists a non-trivial $p \in P_m(K_n)$ such that

$$(f - p_f)(t)\, p(t) \geqslant 0, \qquad t \in E_{f-p_f, [a, b]}. \tag{46}$$

Using (4), if $n > m+1$ and the same argumentation as in the proof of Theorem 3, (ii) $\Rightarrow$ (i) in Section 4, we obtain that p has only a finite number of zeros (counting multiplicities) in $[a, b)$. Since n is even, $N_{[a, b)}(p) \leqslant n$. Therefore, the cardinality of each set $\{t_1^*, ..., t_r^*\}$ with $t_1^* < \cdots < t_r^* < t_1^* + (b-a)$ such that

$$\sigma^* (-1)^k\, p(t_k^*) \geqslant 0, \qquad k = 1, ..., r \tag{47}$$

where $\sigma^* \in \{-1, 1\}$, is at most $n+1$. If

$$\mathscr{A}(f - p_f)|_{[a, b]} \geqslant n+2$$

then (46) implies (47) for a suitable $\sigma^* \in \{-1, 1\}$ with $r \geqslant n+2$, which is a contradiction. Therefore and by (5), we have to consider the case

$$\mathscr{A}(f - p_f)|_{[\xi, \xi + (b-a)]} = n+1$$

where $\xi \in [a, b)$, i.e., there exists $\xi^* \in [a, b)$ such that $f - p_f$ has exactly n alternation intervals $I_k = [\alpha_k, \beta_k]$, $k = 1, ..., n$, in $[\xi^*, \xi^* + (b-a)]$, where $I_{k^*}^0 \neq \varnothing$ for at least one $k^* \in \{1, ..., n\}$. By assumption, there exists a NI-set $\{t_1, ..., t_n\}$ for $P_m(K_n)$ such that $t_k \in I_k$, $k = 1, ..., n$, and $t_k \in I_k^0$ whenever $I_k^0 \neq \varnothing$. Since $I_{k^*}^0 \neq \varnothing$, each set $\{t_1^*, ..., t_n^*\}$ with $t_1^* < \cdots < t_n^* < t_1^* + (b-a)$ such that $t_k^* \in [\beta_k, \alpha_{k+1}]$, $k = 1, ..., n$, differs from $\{t_1, ..., t_n\}$, and obviously $t_k \leqslant t_k^* \leqslant t_{k+1}$, $k = 1, ..., n$. By Theorem 15, each such set $\{t_1^*, ..., t_n^*\}$ is an interpolation set for $P_m(K_n)$. In the following, we show that the assumption (46) leads to a NI-set $\{t_1^*, ..., t_n^*\}$ for $P_m(K_n)$ such that $t_k^* \in [\beta_k, \alpha_{k+1}]$, $k = 1, ..., n$, which is a contradiction.

Since there exists $\sigma \in \{-1, 1\}$ such that

$$\begin{aligned} &\sigma(-1)^k (f - p_f)(t) = \|f - p_f\|_\infty, \\ &t \in E_{f-p_f, [\xi^*, \xi^* + (b-a)]} \cap I_k, \quad k = 1, ..., n \end{aligned} \tag{48}$$

we have by (46),

$$\sigma(-1)^k p(\alpha_k) \geqslant 0 \qquad \text{and} \qquad \sigma(-1)^k p(\beta_k) \geqslant 0, \qquad k = 1, ..., n.$$

Hence, there exist $\tilde{t}_k \in [\beta_k, \alpha_{k+1}]$, $k = 1, ..., n$, such that $p(\tilde{t}_k) = 0$, $k = 1, ..., n$.

Case 1. $\tilde{t}_1 < \cdots < \tilde{t}_n < \tilde{t}_1 + (b-a)$.
Since (4), it follows that

$$card((x_i, x_{i+m+j}) \cap \tilde{T}) \geqslant j, \qquad j = 0, ..., n-m-1, \quad i \in \mathbb{Z}$$

where $\tilde{T} = \{\tilde{t}_{k+ln} = \tilde{t}_k + l(b-a)\colon\ k = 1, ..., n, \ l \in \mathbb{Z}\}$. We conclude that $\{t_1^*, ..., t_n^*\} = \{\tilde{t}_1, ..., \tilde{t}_n\}$ is a NI-set for $P_m(K_n)$.

Case 2. There exist $r \geqslant 1$ and $\mathscr{K}_0 = \{k_1, ..., k_r\} \subseteq \{1, ..., n\}$, such that

$$\tilde{t}_k = \alpha_{k+1} = \beta_{k+1} = \tilde{t}_{k+1}, \qquad k \in \mathscr{K}_0.$$

We choose r maximal. We may assume that each $\tilde{t}_k$ with $k \in \mathscr{K}_0$ is a zero of p where p does not change sign (i.e., $p(\tilde{t}_k - \delta)\, p(\tilde{t}_k + \delta) > 0$, $\delta > 0$), since otherwise an induction argument shows that one could choose $\tilde{t}_k < \tilde{t}_{k+1}$. Moreover, since $N_{[a,b)}(p) \leqslant n$, it follows that if $k \in \mathscr{K}_0$, then $k-1$, $k+2 \notin \mathscr{K}_0$. Set $\mathscr{K}_1 = \{k_1 + 1, ..., k_r + 1\}$. We distinguish between the cases $m \geqslant 2$ and $m = 1$.

Case 2a. $m \geqslant 2$.

Since $N_{[a,b)}(p) \leqslant n$, it follows that $\tilde{t}_k$, $k \in \mathscr{K}_0$ are exactly the double zeros of p in $[\tilde{t}_1, \tilde{t}_1 + (b-a))$ and $\tilde{t}_k$, $k \in \{1, ..., n\} \backslash (\mathscr{K}_0 \cup \mathscr{K}_1)$ are exactly the simple zeros of p in $[\tilde{t}_1, \tilde{t}_1 + (b-a))$ and there does not exist a zero of p in $[\tilde{t}_1, \tilde{t}_1 + (b-a)) \backslash \{\tilde{t}_1, ..., \tilde{t}_n\}$. By (46) and (48), it follows that

$$\sigma(-1)^{k+1} p(t) > 0, \qquad t \in (\tilde{t}_k, \tilde{t}_{k+1}), \qquad k = 1, ..., n. \tag{49}$$

(Note that $(\tilde{t}_k, \tilde{t}_{k+1}) = \varnothing$, if $k \in \mathscr{K}_0$.) By Lemma 13, we have for $n \geqslant m$

$$card((x_i, x_{i+m+j}) \cap \tilde{T}) \geqslant j+1, \qquad j = 0, ..., n-m, \quad i \in \mathbb{Z}$$

where $\tilde{T}=\{\tilde{t}_{k+ln}=\tilde{t}_k+l(b-a): k=1, ..., n, \ l\in\mathbb{Z}\}$. Set $\hat{T}=\{\hat{t}_{k+ln}=\tilde{t}_k+l(b-a): k\in\{1, ..., n\}\backslash(\mathcal{K}_0\cup\mathcal{K}_1), l\in\mathbb{Z}\}$. Thus, if $n-2r>m$,

$$card((x_i, x_{i+m+j})\cap\hat{T})\geqslant j+1-2r, \qquad j=2r-1, ..., n-m, \quad i\in\mathbb{Z}.$$

By Lemma 21, there exists $\hat{p}\in P_m(K_n)$ having exactly the set of zeros $\{\tilde{t}_k: k\in\{1, ..., n\}\backslash(\mathcal{K}_0\cup\mathcal{K}_1)\}$ in $[\tilde{t}_1, \tilde{t}_1+(b-a))$ and all of these zeros are simple. Therefore, by replacing $\hat{p}$ by $-\hat{p}$ if necessary, we have

$$\sigma(-1)^k\,\hat{p}(t)>0, \qquad t\in(\tilde{t}_{k-1}, \tilde{t}_{k+2}), \quad k\in\mathcal{K}_0. \tag{50}$$

Now choose $\delta>0$ such that $\tilde{t}_k-\delta>\beta_k$, $\tilde{t}_k+\delta<\alpha_{k+2}$, $k\in\mathcal{K}_0$, and fix $\varepsilon>0$ such that

$$\sigma(-1)^k\,(p(\tilde{t}_k-\delta)-\varepsilon\hat{p}(\tilde{t}_k-\delta))>0, \qquad k\in\mathcal{K}_0 \tag{51}$$

and

$$\sigma(-1)^k\,(p(\tilde{t}_k+\delta)-\varepsilon\hat{p}(\tilde{t}_k+\delta))>0, \qquad k\in\mathcal{K}_0. \tag{52}$$

Set $p^*=p-\varepsilon\hat{p}\in P_m(K_n)$. Obviously, $p^*\not\equiv 0$. By (50), we have

$$\sigma(-1)^k\,p^*(\tilde{t}_k)=(-1)^{k+1}\,\sigma\varepsilon\hat{p}(\tilde{t}_k)<0, \qquad k\in\mathcal{K}_0.$$

By (51) (respectively (52)), it follows that there exist $t_k^*\in(\tilde{t}_k-\delta, \tilde{t}_k)\subseteq[\beta_k, \alpha_{k+1}]$ (respectively $t_{k+1}^*\in(\tilde{t}_k, \tilde{t}_k+\delta)\subseteq[\beta_{k+1}, \alpha_{k+2}]$), $k\in\mathcal{K}_0$, such that $p^*(t_k^*)=p^*(t_{k+1}^*)=0$. In addition, set $t_k^*=\tilde{t}_k$, $k\in\{1, ..., n\}\backslash(\mathcal{K}_0\cup\mathcal{K}_1)$. Thus, $p^*(t_k^*)=0$, $k\in\{1, ..., n\}\backslash(\mathcal{K}_0\cup\mathcal{K}_1)$ and $t_1^*<\cdots<t_n^*<t_1^*+(b-a)$. By (4) and $t_k^*\in[\beta_k, \alpha_{k+1}]$, $k=1, ..., n$, we have for $n>m+1$

$$card((x_i, x_{i+m+j})\cap T^*)\geqslant j, \qquad j=1, ..., n-m-1, \qquad i\in\mathbb{Z}$$

where $T^*=\{t_{k+ln}^*=t_k^*+l(b-a): k=1, ..., n, l\in\mathbb{Z}\}$. Hence, as in the proof of Lemma 11, it follows that p^* has exactly the set of zeros $\{t_1^*, ..., t_n^*\}$ in $[t_1^*, t_1^*+(b-a))$. Therefore, $\{t_1^*, ..., t_n^*\}$ is a NI-set for $P_m(K_n)$ with $t_k^*\in[\beta_k, \alpha_{k+1}]$, $k=1, ..., n$.

Case 2b. $m=1$.

In this case, $\{\tilde{t}_k: k\in\mathcal{K}_0\}\subseteq K_n$, and it is easily seen that there does not exist a zero of p in $[\tilde{t}_1, \tilde{t}_1+(b-a))\backslash\{\tilde{t}_1, ..., \tilde{t}_n\}$. Set $x_{j_k}=\tilde{t}_k$, $k\in\mathcal{K}_0$, and fix $\varepsilon>0$ such that $s_k\in S_1(x_{j_k})$, $k\in\mathcal{K}_0$, determined by $s_k(\tilde{t}_k)=-p(x_{j_k-1})\,|p(x_{j_k-1})|^{-1}\,\varepsilon$, $s_k(x_{j_k-1})=p(x_{j_k-1})$, and $s_k(x_{j_k+1})=p(x_{j_k+1})$ has the zeros

$t_k^* \in (\beta_k, \alpha_{k+1})$, $t_{k+1}^* \in (\beta_{k+1}, \alpha_{k+2})$, $k \in \mathscr{K}_0$. Moreover, set $t_k^* = \tilde{t}_k$, $k \in \{1, ..., n\} \backslash (\mathscr{K}_0 \cup \mathscr{K}_1)$ and consider the $(b-a)$-periodic extension of

$$p^*(t) = \begin{cases} s_k(t), & \text{if} \quad t \in [x_{j_k - 1}, x_{j_k + 1}], \quad k \in \mathscr{K}_0 \\ p(t), & \text{if} \quad t \in [x_{j_1 - 1}, x_{j_1 - 1 + n}) \backslash \bigcup\limits_{k \in \mathscr{K}_0} [x_{j_k - 1}, x_{j_k + 1}], \end{cases}$$

on the real line, which is evidently in $P_1(K_n)$. By an analogue argumentation as above, $\{t_1^*, ..., t_n^*\}$ is a NI-set for $P_1(K_n)$ with $t_k^* \in [\beta_k, \alpha_{k+1}]$, $k = 1, ..., n$.

This completes the proof of Theorem 5. ∎

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